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Quasivarieties of distributive lattices with a quantifier

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Abstract

It is shown that any subvariety V of the variety of bounded distributive lattices with a quantifier, as considered by Cignoli (1991), contains either uncountably or finitely many quasivarieties depending on whether V contains the 4-element bounded Boolean lattice with a simple quantifier. It is also shown that, in the former case, the quasivarieties contained in V form a lattice which fails to satisfy every nontrivial lattice identity while, in the latter case, they form a chain of length $\leqslant 3$.

1. Introduction

A quantifier on a bounded distributive lattice $(L; \vee, \wedge, 0, 1)$ is a (necessarily idempotent) unary operation ∇ on L that satisfies: $\nabla 0 = 0$, $x \wedge \nabla x = x$, $\nabla(x \vee y) = \nabla x \vee \nabla y$, and $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$. A *Q-distributive lattice* $(L; \vee, \wedge, \nabla, 0, 1)$ is a bounded distributive lattice with a quantifier. Q-distributive lattices were introduced by Cignoli [2]. They appear naturally when the operation of complement is dropped from the monadic Boolean algebras of Halmos [5].

A class of algebras of similar type that is closed under homomorphisms, subalgebras, and direct products is called a *variety*. The class $\boldsymbol{Q}$ of all Q-distributive lattices is a variety the subvarieties of which, as shown in Cignoli [2], from an $\omega + 1$ chain under inclusion:

$$
\begin{array}{l}
\boldsymbol{Q}_{00} \subset \\
\boldsymbol{Q}_{10} \subset \boldsymbol{Q}_{01} \subset \\
\boldsymbol{Q}_{20} \subset \boldsymbol{Q}_{02} \subset \boldsymbol{Q}_{11} \subset \\
\boldsymbol{Q}_{30} \subset \boldsymbol{Q}_{03} \subset \boldsymbol{Q}_{12} \subset \boldsymbol{Q}_{21} \subset \\
\boldsymbol{Q}_{40} \subset \boldsymbol{Q}_{04} \subset \boldsymbol{Q}_{13} \subset \boldsymbol{Q}_{22} \subset \boldsymbol{Q}_{31} \subset \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \boldsymbol{Q}
\end{array}
$$

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such that, for each $(p,q)\in\omega\times\omega$, $\mathbf{Q}_{pq}$ is the variety generated by the algebra $Q_{pq}=(B_p\times C_q;\vee,\wedge,\nabla,0,1)$ where B_p is the p-atom Boolean lattice, $C_0=B_0$ and, for $q\geqslant 1$, $C_q=B_q\oplus 1$ where $\oplus$ denotes ordinal sum, and ∇ is the *simple quantifier* on $B_p\times C_q$ given $\nabla 0=0$ and, for $a\neq 0$, $\nabla a=1$. (Observe, in particular, Q_{00} is a 1-element algebra, Q_{p0} is Q-isomorphic to B_p with a simple quantifier, and Q_{0q} is Q-isomorphic to C_q with a simple quantifier.)

A class of algebras of similar type that is closed under isomorphisms, subalgebras, direct products, and ultraproducts is called a *quasivariety* (in particular, every variety is a quasivariety). The quasivarieties contained in a variety $\mathbf{V}$ form a lattice $L(\mathbf{V})$ with respect to inclusion. Our aim is to show that the structure of $L(\mathbf{Q})$ is far more complicated than the lattice structure of the varieties contained in $\mathbf{Q}$.

Our principal result is the following Theorem.

Theorem 1.1. *For a variety $\mathbf{V}$ of Q-distributive lattices the following conditions hold:*

(i) *$L(\mathbf{V})$ is either finite or of cardinality $2^{\aleph_0}$;*

(ii) *$L(\mathbf{V})$ is finite if and only if $L(\mathbf{V})$ is a chain of length $\leqslant 3$ if and only if $\mathbf{V}\subseteq\mathbf{Q}_{01}$;*

(iii) *$L(\mathbf{V})$ has cardinality $2^{\aleph_0}$ if and only if a free lattice with ω free generators is embeddable in $L(\mathbf{V})$ if and only if $\mathbf{Q}_{20}\subseteq\mathbf{V}$.*

The proof of the above theorem is concluded in Section 6 using a method that was developed in [1] (see Section 4 for the necessary details) and a duality for $\mathbf{Q}$ given by Cignoli [2] (see Section 2 for the necessary details). In Section 3, it is shown that $L(\mathbf{Q}_{01})$ is a chain of length 3. To conclude that, for $\mathbf{Q}_{20}\subseteq\mathbf{V}$, $L(\mathbf{V})$ has cardinality $2^{\aleph_0}$ and that a free lattice with ω free generators is embeddable in $L(\mathbf{V})$ we will use a method developed in [1]. The method requires the existence of a family of Q-distributive lattices with certain properties. Such a family is described (via the duality) in Section 5 and that it has the required properties is established in Section 6. In fact, we will show more: the ideal lattice of a free lattice with ω free generators is embeddable in $L(\mathbf{V})$ whenever $\mathbf{Q}_{20}\subseteq\mathbf{V}$. In Section 7, we return to monadic Boolean algebras which also form a variety. Proposition 7.1 gives a description of $L(\mathbf{V})$ for every variety $\mathbf{V}$ of monadic Boolean algebras.

The above theorem reveals an underlying similarity between Q-distributive lattices and pseudocomplemented distributive lattices (see [4, 7, 11]). However, the reason for this is still unknown.

2. Duality

Priestley [9] established that the category of (0, 1)-distributive lattices with (0, 1)-lattice homomorphisms is dually equivalent to a category whose objects are certain partially ordered sets endowed with a topology. Subsequently, Cignoli [2] (cf. [10]) derived an analogous duality for the category of Q-distributive lattices. In this section we recall the details. However, since we need only to consider finite Q-distributive lattices, we may dispense with all topological aspects (since, for finite structures, the topology is always discrete).

Let $\boldsymbol{Q}_{\text{fin}}$ denote the category of finite Q-distributive lattices whose morphisms are $(0,1)$-lattice homomorphisms that preserve ∇. We denote by $\boldsymbol{S}_{\text{fin}}$ the category whose objects $(P;\leqslant,E)$, called *Q-spaces*, consist of all finite partially ordered sets $(P;\leqslant)$ endowed with an equivalence relation E satisfying: for $x,y\in P$, if $x\leqslant y$ and xEz, then $z\leqslant s$ and yEs for some $s\in P$. The morphisms in $\boldsymbol{S}_{\text{fin}}$, called *Q-maps*, are mappings $\varphi:(P;\leqslant,E)\to(P';\leqslant',E')$ such that, for $x,y\in P$, $\varphi(x)\leqslant\varphi(y)$ whenever $x\leqslant y$, $\varphi(x)E'\varphi(y)$ whenever xEy, and $z=\varphi(y)$ for some $y\in[x]E$ whenever z is a maximal element in the equivalence class $[\varphi(x)]E'$.

The concept of a Q-space with an accompanying topology and an appropriately defined Q-map were introduced by Cignoli [2] to form a category $\boldsymbol{S}$ dually equivalent to the category $\boldsymbol{Q}$ of all Q-distributive lattices. Cignoli's definitions of a Q-space and Q-map are equivalent within the class of finite partially ordered sets to the above. The contravariant functors $\mathscr{S}:\boldsymbol{Q}\to\boldsymbol{S}$ and $\mathscr{Q}:\boldsymbol{S}\to\boldsymbol{Q}$ and the pair of natural isomorphisms $\sigma:1_{\boldsymbol{Q}}\cong\mathscr{Q}\mathscr{S}$ and $\varepsilon:1_{\boldsymbol{S}}\cong\mathscr{S}\mathscr{Q}$ that establish a dual equivalence between $\boldsymbol{Q}$ and $\boldsymbol{S}$, but restricted to the full subcategories $\boldsymbol{Q}_{\text{fin}}$ and $\boldsymbol{S}_{\text{fin}}$ of $\boldsymbol{Q}$ and $\boldsymbol{S}$, respectively, are defined as follows:

With each object A of $\boldsymbol{Q}_{\text{fin}}$ is associated a Q-space

$$\mathscr{S}(A)=(S(A);\leqslant,E),$$

where $S(A)$ is the set of all nonzero $\vee$-irreducible elements of A, $\leqslant$ is the reverse of the order induced by A, and, for $a,b\in S(A)$, $(a,b)\in E$ if and only if $\nabla a=\nabla b$. With each morphism $f:A\to A'$ in $\boldsymbol{Q}_{\text{fin}}$ is associated a Q-map $\mathscr{S}(f):S(A')\to S(A)$ such that, for $a\in A'$,

$$\mathscr{S}(f)(a)=\bigwedge(x\in S(A):\ f(x)\geqslant a).$$

With each object $(P;\leqslant,E)$ of $\boldsymbol{S}_{\text{fin}}$ is associated a Q-distributive lattice

$$\mathscr{Q}(P)=(Q(P);\cup,\cap,\nabla,\emptyset,P),$$

where $Q(P)$ is the set of all order filters of P, $\cup$ and $\cap$ are set-theoretical union and intersection, respectively, and, for $U\in Q(P)$, $\nabla U=\bigcup([x]E:\ x\in U)$. With each morphism $\varphi:P\to P'$ in $\boldsymbol{S}_{\text{fin}}$ is associated a Q-homomorphism $\mathscr{Q}(\varphi):Q(P')\to Q(P)$ such that, for $U\in Q(P')$,

$$\mathscr{Q}(\varphi)(U)=\varphi^{-1}(U).$$

The natural isomorphisms σ: $1_{\boldsymbol{Q}_{\text{fin}}}\cong\mathscr{Q}\mathscr{S}$ and ε: $1_{\boldsymbol{S}_{\text{fin}}}\cong\mathscr{S}\mathscr{Q}$ are given by $\sigma(A)(a)=\{x\in Q(A):\ x\leqslant a\}$ and $\varepsilon(P)(x)=\{[y]\in Q(P):\ y\leqslant x\}$.

Proposition 2.1 (Cignoli [2]). *The category $\boldsymbol{Q}_{\text{fin}}$ is dually equivalent to the category $\boldsymbol{S}_{\text{fin}}$. The dual equivalence is given by the pair of contravariant functors $\mathscr{S}:\boldsymbol{Q}_{\text{fin}}\to\boldsymbol{S}_{\text{fin}}$, $\mathscr{Q}:\boldsymbol{S}_{\text{fin}}\to\boldsymbol{Q}_{\text{fin}}$ and the pair of natural isomorphisms σ: $1_{\boldsymbol{Q}_{\text{fin}}}\cong\mathscr{Q}\mathscr{S}$ and ε: $1_{\boldsymbol{S}_{\text{fin}}}\cong\mathscr{S}\mathscr{Q}$.*

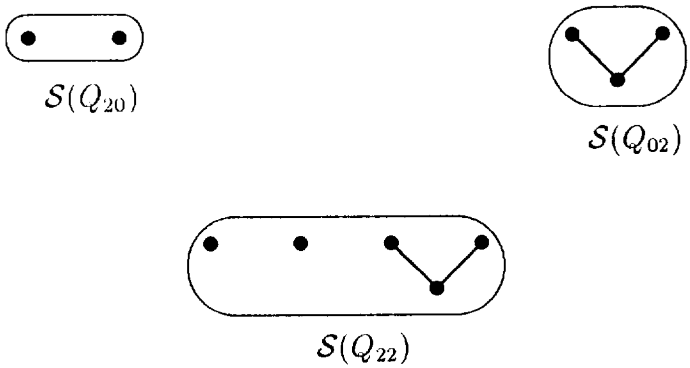

Fig. 1.

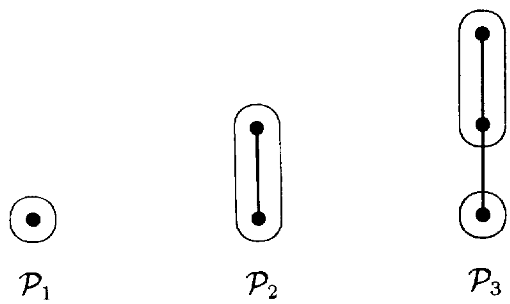

Fig. 2.

Since products in $\mathbf{Q}_{\text{fin}}$ correspond to co-products in $\mathbf{S}_{\text{fin}}$ (which are disjoint unions of Q-spaces) and one-to-one (onto) morphisms in $\mathbf{Q}_{\text{fin}}$ correspond to onto (one-to-one order-isomorphic) morphisms in $\mathbf{S}_{\text{fin}}$, for our purposes it is easier to work in the category $\mathbf{S}_{\text{fin}}$ than the category $\mathbf{Q}_{\text{fin}}$ and we shall do so.

By way of example, $\mathscr{S}(Q_{20})$, $\mathscr{S}(Q_{02})$, and $\mathscr{S}(Q_{22})$ are diagrammed in Fig. 1 where both the underlying partially ordered sets and the equivalence relations defined over them are given. (Since ∇ is the simple quantifier on Q_{20}, Q_{02}, and Q_{22}, each of $\mathscr{S}(Q_{20})$, $\mathscr{S}(Q_{02})$, and $\mathscr{S}(Q_{22})$ have only one equivalence class.) Observe also that $\mathscr{S}(Q_{10})$ and $\mathscr{S}(Q_{01})$ are Q-isomorphic to $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively, of Fig. 2.

3. Quasivarieties in $L(\mathbf{Q}_{01})$

In this section we show that $L(\mathbf{Q}_{01})$ is a 4-element chain.

Lemma 3.1. *The Q-distributive lattice $Q(P)$ of a finite Q-space $(P; \leqslant, E)$ belongs to $\mathbf{Q}_{01}$ if and only if, for each $x \in P$, the partially ordered set $([x]E; \leqslant \restriction [x]E)$ contains exactly one maximal element.*

Proof. *Let* $Q(P) \in \boldsymbol{Q}_{01}$ and suppose that, for some equivalence class X of P, $|M| \geqslant 2$ where M denotes the set of maximal elements of X. Consider the Q-space $(M; \leqslant \restriction M, E \restriction M)$ (where, for notational simplicity, $\leqslant \restriction M$ and $E \restriction M$ denote $\leqslant \restriction M \times M$ and $E \restriction M \times M$, respectively). The identity map from M into P is a Q-map. By the comments following Proposition 2.1, $Q(M)$ is a Q-homomorphic image of $Q(P)$. Since $Q(M) \cong Q_{p0}$ where $p = |M| \geqslant 2$, $Q(P) \notin \boldsymbol{Q}_{01}$.

Conversely, for each equivalence class X in P define $P_X^x = (\{x, m\}; \leqslant \restriction \{x, m\}, E \restriction \{x, m\})$ where m is the maximal element of X and $x \in X$. Clearly, P_X^x is a Q-space, $Q(P_X^x) \cong Q_{10}$ for $x = m$, and $Q(P_X^x) \cong Q_{01}$ otherwise. Since the family of Q-homomorphisms $Q(id): Q(P) \to Q(P_X^x)$ separates the elements of $Q(P)$, it follows, by the remarks after Proposition 2.1, that $Q(P)$ is a subdirect product of the family of Q-distributive lattices $Q(P_X^x)$, where X and x are as above. In particular, $Q(P) \in \boldsymbol{Q}_{01}$. $\square$

For a Q-space $(P; \leqslant, E)$ and $P' \subseteq P$, $(P'; \restriction P', E \restriction P')$ is a Q-subspace providing $id: P' \to P$ is a Q-map; it is a *proper* Q-subspace providing $P' \subset P$. A Q-space $(P; \leqslant, E)$ is *critical* if, for some $x \in P$, no proper Q-subspace of P containing x is a Q-map image of P. The next lemma will characterize those critical Q-spaces whose associated Q-distributive lattices belong to $\boldsymbol{Q}_{01}$.

Let $\mathscr{P}_1 = (1; \leqslant, 1 \times 1)$, $\mathscr{P}_2 = (2; \leqslant, 2 \times 2)$, and $\mathscr{P}_3 = (3; \leqslant, (1 \times 1) \cup (\{1,2\} \times \{1,2\}))$ where $n = \{0, 1, \ldots, n-1\}$ (see Fig. 2). It is readily seen that $\mathscr{P}_1$, $\mathscr{P}_2$, and $\mathscr{P}_3$ are critical Q-spaces. By Lemma 3.1, $Q(\mathscr{P}_1)$, $Q(\mathscr{P}_2)$, and $Q(\mathscr{P}_3)$ belong to $\boldsymbol{Q}_{01}$.

Lemma 3.2. *A Q-space $(P; \leqslant, E)$ whose Q-distributive lattice $Q(P)$ belongs to $\boldsymbol{Q}_{01}$ is critical if and only if $P \cong \mathscr{P}_1$, $\mathscr{P}_2$, or $\mathscr{P}_3$.*

Proof. Let $(P; \leqslant, E)$ be a critical Q-space and $Q(P) \in \boldsymbol{Q}_{01}$. Define $\preceq$ on P/E by $[x]E \preceq [y]E$ if and only if $x' \leqslant y'$ for some $x'Ex$ and $y'Ey$. It follows from the definition of a Q-space that $(P/E; \preceq)$ is a partially ordered set. There are two cases to consider.

Case 1: There exists an equivalence class $[x]E$ that is minimal with respect to $\preceq$ for which $|[x]E| \geqslant 2$.

By Lemma 3.1, $[x]E$ contains a unique maximal element m_x. Define $\varphi: P \to \mathscr{P}_2$ by $\varphi(y) = 1$ if $y = m_x$ or $y \notin [x]E$ and $\varphi(y) = 0$ otherwise. Clearly, φ is a Q-map and, since $|[x]E| \geqslant 2$, φ is onto. By Lemma 3.1, every element of P is contained in a Q-subspace that is a Q-map image of $\mathscr{P}_2$. Since P is critical, it follows that φ is one-to-one and, hence, $P \cong \mathscr{P}_2$.

Case 2: For every equivalence class $[x]E$ that is minimal with respect to $\preceq$, $|[x]E| = 1$.

There are two subcases.

Subcase 2(a): For every $y \in P$, $|[y]E| = 1$. It is readily seen that $P \cong \mathscr{P}_1$.

Subcase 2(b). For some $y \in P$, $|[y]E| \geqslant 2$. Let m_y denote the maximal element of $[y]E$ and define $\varphi: P \to \mathscr{P}_3$ by $\varphi(z) = 2$ if $z = m_y$ or $z \nleqslant m_y$, $\varphi(z) = 1$ if $z \neq m_y$ and

$z \in [y]E$, and $\varphi(z)=0$ otherwise. It is not hard to see that φ is a well-defined Q-map. Moreover, under the hypothesis of Case 2, it is readily seen that every element of P is contained in a Q-subspace of P that is a Q-map image of $\mathscr{P}_3$. Since P is critical, it follows that $P \cong \mathscr{P}_3$. $\square$

A finite algebra is *critical* if it is not isomorphic to a subdirect product of any family of its proper subalgebras. Critical algebras are of interest since any non-trivial locally finite quasivariety is generated by its critical members. To see this notice that the quasivariety generated by a finite algebra A coincides with the quasivariety generated by the subalgebras of A that are critical and that a locally finite quasivariety is generated by its finite algebras. In particular, $\boldsymbol{Q}_{01}$ and its subquasivarieties are locally finite and, hence, each one of them is generated by its critical members.

Given the remarks following Proposition 2.1, it is not hard to see that a Q-space P is critical if and only if the Q-distributive lattice $Q(P)$ is critical.

Theorem 3.3. *The quasivarieties contained in $\boldsymbol{Q}_{01}$ form a* 4*-element chain $\boldsymbol{T} \subset \boldsymbol{Q}_{\mathscr{P}_1} \subset \boldsymbol{Q}_{\mathscr{P}_3} \subset \boldsymbol{Q}_{\mathscr{P}_2} = \boldsymbol{Q}_{01}$ where $\boldsymbol{T}$ denotes the quasivariety of all* 1*-element algebras and, for $1 \leqslant i \leqslant 3$, $\boldsymbol{Q}_{\mathscr{P}_i}$ denotes the quasivariety generated by $Q(\mathscr{P}_i)$.*

Proof. By Lemma 3.2, it is enough to observe that $\mathscr{P}_1$ is a Q-map image of both $\mathscr{P}_2$ and $\mathscr{P}_3$, $\mathscr{P}_3$ is a Q-map image of two copies of $\mathscr{P}_2$, there are elements of $\mathscr{P}_2$ and $\mathscr{P}_3$ which lie in no Q-map image of $\mathscr{P}_1$, and there exists an element of $\mathscr{P}_3$ which lies in no Q-map image of $\mathscr{P}_2$. $\square$

4. An embedding of the ideal lattice of a free lattice

For a set M of similar algebras, let $V_{\mathrm{Q}}(M)$ denote the quasivariety generated by M.

Let $P_{\mathrm{fin}}(\omega)$ denote the set of all finite subsets of ω and consider an infinite family $(A_W\colon W \in P_{\mathrm{fin}}(\omega))$ of finite algebras of similar type that satisfy the following conditions, where X, Y, and $Z \in P_{\mathrm{fin}}(\omega)$:

(P1) $A_\emptyset$ is a trivial algebra;

(P2) if $X = Y \cup Z$, then $A_X \in V_{\mathrm{Q}}(\{A_Y, A_Z\})$;

(P3) if $X \neq \emptyset$ and $A_X \in V_{\mathrm{Q}}(\{A_Y\})$, then $X = Y$;

(P4) if A_X is a subalgebra of $B \times C$ for finite B, $C \in V_{\mathrm{Q}}(\{A_W\colon W \in P_{\mathrm{fin}}(\omega)\})$, then there exist Y and Z with $A_Y \in V_{\mathrm{Q}}(\{B\})$, $A_Z \in V_{\mathrm{Q}}(\{C\})$, and $X = Y \cup Z$.

To establish that $L(\boldsymbol{Q}_{20})$ has cardinality $2^{\aleph_0}$ and that a free lattice with ω free generators is embeddable in it we will need to refer to the following proposition:

Proposition 4.1 [1]. *If $\boldsymbol{L}$ is a quasivariety of algebras of finite type that contains an infinite family of finite algebras satisfying* (P1)–(P4), *then the ideal lattice of a free lattice with ω generators is embeddable in $L(\boldsymbol{L})$. In particular, $L(\boldsymbol{L})$ fails every nontrivial lattice identity and is of cardinality $2^{\aleph_0}$.* $\square$

5. Q-spaces associated with graphs

To employ Proposition 4.1, it will be necessary to determine a suitable family $(A_W\colon W\in P_{\mathrm{fin}}(\omega))$ of finite algebras in $\boldsymbol{Q}_{20}$. In this section, we shall define such a family in terms of their Q-spaces and establish some basic properties of those spaces. That the family actually lies in $\boldsymbol{Q}_{20}$ and satisfies (P1)–(P4) will be verified in the next section.

The construction of Q-spaces given below will make use of some rather special graphs. A *graph* $G=(V,E)$ is a set V whose elements are called *vertices* together with a collection E of 2-element subsets of V called *edges*. A graph G is *connected* if, for every $x,y\in V$, there exists a sequence $z_0,\dots,z_n$ of elements of V such that $x=z_0$, $y=z_n$, and $\{z_i,z_{i+1}\}\in E$ for all $i<n$. A mapping φ between graphs G and G' is *compatible* providing that, for $x,y\in V$, $\{\varphi(x),\varphi(y)\}\in E'$ whenever $\{x,y\}\in E$. To show that the given Q-spaces have the required properties, we need the existence of an infinite family of nonisomorphic finite graphs that are connected, for which there are no compatible maps between distinct members of the family, and for which the only compatible mappings from a member of the family to itself are onto. The existence of such a family, denoted $(G_i=(V_i,E_i)\colon i<\omega)$, follows from Hedrlín and Sichler [6]. For some notational ease and with no loss in generality, we will assume that, for distinct $i,j<\omega$, G_i and G_j have no vertices in common.

For $W\in P_{\mathrm{fin}}(\omega)$ with $W\neq\emptyset$, define a Q-space $(P(W);\leqslant,E(W))$ as follows:

$$P(W)=\{a,b\}\cup\bigcup(D_i\colon i\in W)\cup\bigcup(\{e_x,e_y\}\colon e=\{x,y\}\in E_i \text{ and } i\in W),$$

where a and b are two fixed elements and $D_i=V_i\times\{-j,j\colon 1\leqslant j\leqslant 5\}$.

The equivalence relation $E(W)$ is determined by the following partition on $P(W)$:

$$\{\{a\},\{b\}\}\cup\{\{(x,-j),(x,j)\}\colon x\in V_i, i\in W, \text{ and } 1\leqslant j\leqslant 5\}$$
$$\cup\{\{e_x,e_y\}\colon e=\{x,y\}\in E_i \text{ and } i\in W\}.$$

The partial order $\leqslant$ on $P(W)$ is given in two parts. Firstly, for $x\in V_i$ and $i\in W$, $(x,\pm j)<(x,\pm k)$ for $j=2$ or 4 and $k=j\pm 1$ with two exceptions; namely, $(x,2)\nless(x,3)$ and $(x,4)\nless(x,5)$. Secondly, for $i\in W$ and $e=\{x,y\}\in E_i$, $(x,\pm j)$ and $(y,\pm j)<e_x$ and e_y for $j=2$ or 4 with two exceptions; namely, $(x,4)\nless e_x$ and $(y,4)\nless e_y$. See Figs. 3 and 4 which diagram the partial order and the equivalence relation on relevant parts of $P(W)$.

For $W\in P_{\mathrm{fin}}(\omega)$, it is not hard to see that $P(W)$ is a connected poset in which every element is either maximal or minimal. The maximal elements are precisely those of the form $(x,\pm j)$, for $j=1$, 3, or 5, $x\in V_i$, and $i\in W$, or of the form e_x for $e=\{x,y\}\in E_i$ and $i\in W$. The minimal elements are precisely a, b, or those of the form $(x,\pm j)$ for $j=2$ or 4, $x\in V_i$, and $i\in W$. There are just two equivalence classes of cardinality one: the equivalence class that contains a and the equivalence class that contains b. All other equivalence classes have cardinality two: the equivalence classes of the form $\{(x,-j),(x,j)\}$ for $1\leqslant j\leqslant 5$, $x\in V_i$, and $i\in W$ and of the form $\{e_x,e_y\}$ for $e=\{x,y\}\in E_i$ and $i\in W$.

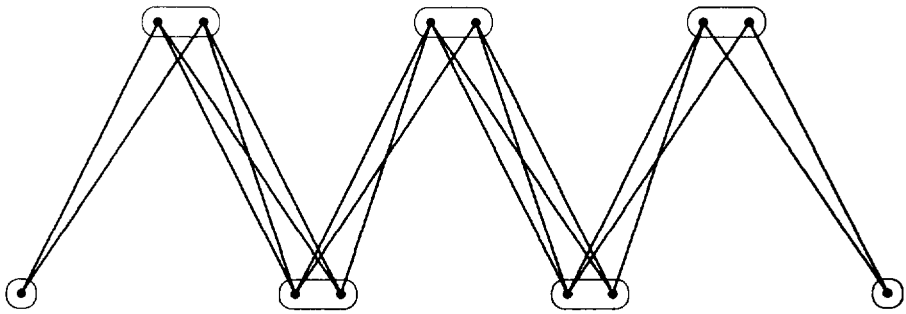

Fig. 3.

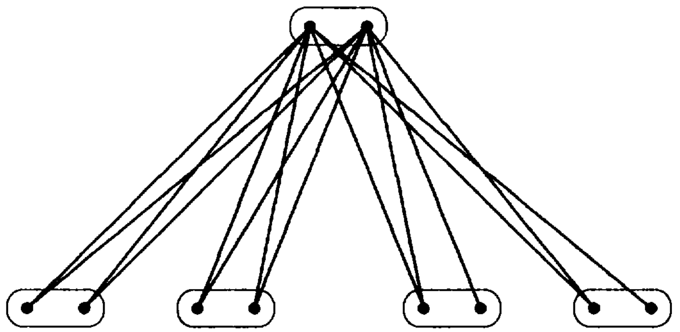

Fig. 4.

As such, it is readily verified that, for r, s, $t \in P(W)$, if $r \leqslant s$ and $rE(W)t$, then $t \leqslant u$ and $sE(W)u$ for some $u \in P(W)$. We conclude the following lemma.

Lemma 5.1. *For $W \in P_{\text{fin}}(\omega)$, $(P(W); \leqslant, E(W))$ is a Q-space.*

Essentially, Lemmas 5.2 and 5.3 show that, for X, $Y \in P_{\text{fin}}(\omega)$, there exist elements of $P(Y)$ which ensure that $\varphi(a) = a$ and $\varphi(b) = b$ whenever they lie in the image of any Q-map $\varphi: P(X) \to P(Y)$. Should $\varphi(a) = a$ and $\varphi(b) = b$, $X \subseteq Y$ follows from the choice of the family $(G_i\colon i < \omega)$: an important step in the verification of (P3) and (P4) in Section 6.

Lemma 5.2. *For X, $Y \in P_{\text{fin}}(\omega)$, let $\varphi: P(X) \to P(Y)$ be a Q-map. If $\varphi(a) = \varphi(b)$, then either*

(i) $\varphi(a) = a$ *and* $\varphi(P(X)) \subseteq \{a\} \cup \bigcup (V_j \times \{-k, k\}\colon j \in Y$ *and* $k = 1$ *or* $2) \cup \bigcup (\{e_x, e_y\}\colon e = \{x, y\} \in E_j$ *and* $j \in Y)$, *or*

(ii) $\varphi(a) = b$ *and* $\varphi(P(X)) \subseteq \{b\} \cup \bigcup (V_j \times \{-5, 5\}\colon j \in Y)$.

Proof. Assume $\varphi(a)=\varphi(b)$. By definition, for $W\in P_{\mathrm{fin}}(\omega)$ and $x\in P(W)$, $\{x\}$ is in the partition determined by $E(W)$ exactly when $x=a$ or b and, otherwise, $[x]E(W)$ is a 2-element antichain. In particular, each element of $[x]E(W)$ is maximal. Since φ is a Q-map, it follows that, for $x\in P(X)$, $[\varphi(x)]E(Y)=\varphi([x]E(X))$. Hence, $\varphi(a)=\varphi(b)=a$ or b.

Case 1: $\varphi(a)=\varphi(b)=a$.

Since φ is order-preserving, $\varphi(P(X))\subseteq\{a\}\cup\bigcup(V_j\times\{-k,k\}\colon j\in Y \text{ and } 1\leqslant k\leqslant 3)\cup\bigcup(\{e_x,e_y\}\colon e=\{x,y\}\in E_j \text{ and } j\in Y)$. Suppose that $\varphi(P(X))\cap(V_j\times\{-3,3\})\neq\emptyset$ for some $j\in Y$. There are two possibilities. Either there exist $i\in X$, $x\in V_i$, and $y\in V_j$ such that $\varphi(\{(x,-3),(x,3)\})=\{(y,-3),(y,3)\}$ or there exist $i\in X$, $e=\{x,z\}\in E_i$ and $y\in V_j$ such that $\varphi(\{e_x,e_z\})=\{(y,-3),(y,3)\}$. In either case

$$\varphi(\{(x,-2),(x,2)\})=\varphi(\{(x,-4),(x,4)\})=\{(y,-2),(y,2)\}.$$

However, neither $\varphi(\{(x,-3),(x,3)\})=\{(y,-3),(y,3)\}$ and $\varphi(\{(x,-4),(x,4)\})=\{(y,-2),(y,2)\}$ nor $\varphi(\{e_x,e_z\})=\{(y,-3),(y,3)\}$ and $\varphi(\{(x,-2),(x,2)\})=\{(y,-2),(y,2)\}$ are possible for an order-preserving map φ. Thus, as required, $\varphi(P(X))\cap(V_j\times\{-3,3\})=\emptyset$ for all $j\in Y$.

Case 2. $\varphi(a)=\varphi(b)=b$.

Similarly, since φ is order-preserving, $\varphi(P(X))\subseteq\{b\}\cup\bigcup(V_j\times\{-k,k\}\colon j\in Y$ and $3\leqslant k\leqslant 5)\cup\bigcup(\varphi(\{e_x,e_y\}\colon e=\{x,y\}\in E_j$ and $j\in Y)$. If, for some $i\in X$, $x\in V_i$, $j\in Y$, and $y\in V_j$, $\varphi(\{(x,-2),(x,2)\})=\{(y,-4),(y,4)\}$, then it must be the case that $\varphi(\{(x,-1),(x,1)\})=\{(y,-5),(y,5)\}$, which is not possible for an order-preserving map φ. Thus, for $i\in X$ and $x\in V_i$, $\varphi(\{(x,-2),(x,2)\})\in\{b\}\cup\bigcup(V_j\times\{-5,5\}\colon j\in Y)$ and, consequently,

$$\varphi(P(X)\setminus\bigcup(V_i\times\{-4,4\}\colon i\in X))\subseteq\{b\}\cup\bigcup(V_j\times\{-5,5\}\colon j\in Y).$$

If, for some $i\in X$ and $x\in V_i$, $\varphi(\{(x,-4),(x,4)\})\notin\{b\}\cup\bigcup(V_j\times\{-5,5\}\colon j\in Y)$, then, for some $j\in Y$ and $y\in V_j$, $\varphi(\{(x,-4),(x,4)\})=\{(y,-4),(y,4)\}$ and $\varphi(\{(x,-3),(x,3)\})=\{(y,-5),(y,5)\}$, which is not possible for an order-preserving map φ. Thus, as required, $P(X)\subseteq\{b\}\cup\bigcup(V_j\times\{-5,5\}\colon j\in Y)$. □

Lemma 5.3. *For $X, Y\in P_{\mathrm{fin}}(\omega)$, let $\varphi\colon P(X)\to P(Y)$ be a Q-map. If $\varphi(a)\neq\varphi(b)$, then* (i) $\varphi(a)=a$, (ii) $\varphi(b)=b$, (iii) $X\subseteq Y$, *and* (iv) *for* $i\in X$, $\varphi(D_i)=D_i$ *and* $\varphi(\{e_x,e_y\colon e=\{x,y\}\in E_i\})=\{e_x,e_y\colon e=\{x,y\}\in E_i\}$.

Proof. If $\varphi(a)\neq\varphi(b)$, then, arguing as before, either $\varphi(a)=a$ and $\varphi(b)=b$ or $\varphi(a)=b$ and $\varphi(b)=a$. If $\varphi(a)=b$ and $\varphi(b)=a$, then, for each $x\in V_i$ and $i\in X$, there exist some $y\in V_j$ and $j\in Y$ such that $\varphi(\{(x,-1),(x,1)\})=\{(y,-5),(y,5)\}$ and $\varphi(\{(x,-2),(x,2)\})=\{(y,-4),(y,4)\}$, which is not possible for an order-preserving map φ. Thus, $\varphi(a)=a$ and $\varphi(b)=b$.

Further, for $x\in V_i$ and $i\in X$,

$$\varphi(\{(x,-3),(x,3)\})\in\{\{(y,-3),(y,3)\}:\ y\in V_j \text{ and } j\in Y\}$$
$$\cup\{\{e_y,e_z\}:\ e=\{y,z\}\in E_j \text{ and } j\in Y\}.$$

Suppose $\varphi(\{(x,-3),(x,3)\})=\{e_y,e_z\}$ for some $x\in V_i$, $i\in X$, $e=\{y,z\}\in E_j$, and $j\in Y$. Either $\varphi(\{(x,-4),(x,4)\})=\{(y,-4),(y,4)\}$ or $\varphi(\{(x,-4),(x,4)\})=\{(z,-4),(z,4)\}$, neither of which is possible for an order-preserving map φ. Thus,

$$\varphi(\{(x,-3),(x,3)\})\in\{\{(y,-3),(y,3)\}:\ y\in V_j \text{ and } j\in Y\}.$$

Consequently, for $x\in V_i$ and $i\in X$, there exists $y\in V_j$ and $j\in Y$ such that, for $1\leqslant k\leqslant 5$, $\varphi(\{(x,-k),(x,k)\})=\{(y,-k),(y,k)\}$.

By the above, if G and H are the (necessarily disjoint) unions of the graphs $(G_i\colon i\in X)$ and $(G_j\colon j\in Y)$, respectively, then the mapping $\psi\colon G\to H$ given by $\psi(x)=y$ where, for $1\leqslant k\leqslant 5$, $\psi(\{(x,-k),(x,k)\})=\{(y,-k),(y,k)\}$ is well-defined. To see that ψ is a compatible map, let $\{x,x'\}$ be an edge of G, $\psi(x)=y$, and $\psi(x')=y'$. Since $\varphi(\{(x,-k),(x,k)\})=\{(y,-k),(y,k)\}$ and $\varphi(\{(x',-k),(x',k)\})=\{(y',-k),(y',k)\}$ for $k=2$ and 4, either $\varphi(\{e_x,e_{x'}\})=\{e_y,e_{y'}\}$ or $y=y'$ and $\varphi(\{e_x,e_{x'}\})=\{(y,-3),(y,3)\}$. Since the latter is not consistent with an order-preserving map φ, it follows that $\varphi(\{e_x,e_{x'}\})=\{e_y,e_{y'}\}$ and, as required, $\{\psi(x),\psi(x')\}=\{y,y'\}$ is an edge of H. By choice, G_i is a connected graph for each $i\in X$. Consequently, there exists $j\in Y$ such that $\psi\restriction G_i\colon G_i\to G_j$ is a compatible mapping which, again by choice, is only possible providing $j=i$. It follows that $X\subseteq Y$. Furthermore, since the only compatible mappings from G_i to itself are onto, $\varphi(D_i)=D_i$ and $\varphi(\{e_x,e_y\colon e=\{x,y\}\in E_i\})=\{e_x,e_y\colon e=\{x,y\}\in E_i\}$. $\square$

6. Quasivarieties in $L(\mathbf{Q}_{20})$

We now define a family $(A_W\colon W\in P_{\mathrm{fin}}(\omega))$ of finite Q-distributive lattices. Let $A_\emptyset$ denote a trivial Q-distributive lattice and, for $\emptyset\neq W\in P_{\mathrm{fin}}(\omega)$, let A_W denote the Q-distributive lattice associated with $P(W)$, namely, $Q(P(W))$.

Lemma 6.1. *For each $W\in P_{\mathrm{fin}}(\omega)$, A_W belongs to $\mathbf{Q}_{20}$.*

Proof. Clearly, $A_\emptyset\in\mathbf{Q}_{20}$. For each equivalence class X in $P(W)$, where $\emptyset\neq W\in P_{\mathrm{fin}}(\omega)$, let $P_X=(X;\leqslant\restriction X, E\restriction X)$. There are two possibilities for the Q-space P_X: $Q(P_X)\cong Q_{10}$ if $|X|=1$ and $Q(P_X)\cong Q_{20}$ if $|X|=2$. Since the family of Q-homomorphisms $Q(id)\colon Q(P(W))\to Q(P_X)$ separates the elements of $Q(P(W))$, it follows that $Q(P(W))$ is a subdirect product of the family of Q-distributive lattices $Q(P_X)$, where X is defined as above. Thus, for $\emptyset\neq W\in P_{\mathrm{fin}}(\omega)$, $Q(P(W))$ belongs to $\mathbf{Q}_{20}$. $\square$

The remainder of this section is devoted to showing that the family of finite Q-distributive lattices $(A_W\colon W\in P_{\mathrm{fin}}(\omega))$ satisfies the postulates (P1)–(P4) of

Proposition 4.1. This together with the preceding lemma will enable us to conclude that the ideal lattice of a free lattice with ω free generators is embeddable in $L(\mathbf{Q}_{20})$.

Lemma 6.2. *The family* $(A_W\colon W\in P_{\text{fin}}(\omega))$ *satisfies* (P1)–(P3).

Proof. (P1) is satisfied by definition.

To see that (P2) is satisfied, it is sufficient to let $X=Y\cup Z$ where neither Y nor Z are empty. If $P(Y)+P(Z)$ denotes the disjoint union of $P(Y)$ and $P(Z)$, then there is a natural Q-map $\varphi\colon P(Y)+P(Z)\to P(X)$ which, since $X=Y\cup Z$, is onto. By the remarks following Proposition 2.1, it follows that A_X belongs to $V_Q(\{A_Y,A_Z\})$. To verify $P(3)$, let $\emptyset\neq X$, $Y\in P_{\text{fin}}(\omega)$ and $A_X\in V_Q(\{A_Y\})$. In particular, A_X is isomorphic to a subalgebra of A_Y^m for some finite m. By the remarks following Proposition 2.1, it follows that there is an onto Q-map $\varphi\colon\sum(P_k(Y)\colon k<m)\to P(X)$ where $\sum(P_k(Y)\colon k<m)$ denotes the Q-space obtained from the disjoint union of the family of Q-spaces $(P_k(Y)\colon k<m)$ each of which is a copy of $P(Y)$. Choose $i\in X$ and $x\in V_i$. Since φ is onto, $(x,3)\in\varphi(P_k(Y))$ for some $k<m$. By Lemma 5.2, $\varphi(a)\neq\varphi(b)$ where a and b are identified with their respective copies in $P_k(Y)$. By Lemma 5.3, $Y\subseteq X$. Furthermore, since $(x,3)\in D_i$, $(x,3)\in\varphi(D_j)$ for some D_j in $P_k(Y)$ with $j\in Y$. By Lemma 5.3, $\varphi(D_j)=D_j$ and it follows that $i=j\in Y$. Consequently, $X\subseteq Y$ and $X=Y$, as required. □

Lemma 6.3. *The family* $(A_W\colon W\in P_{\text{fin}}(\omega))$ *satisfies* (P4).

Proof. Let $X\in P_{\text{fin}}(\omega)$ and A_X be a subalgebra of $B\times C$ for finite B and $C\in V_Q(\{A_W\colon W\in P_{\text{fin}}(\omega)\})$. We must establish the existence of suitable Y and $Z\in P_{\text{fin}}(\omega)$. Since (P4) is readily seen to hold otherwise, assume $X\neq\emptyset$ and that both B and C are nontrivial Q-distributive lattices. By hypothesis and the remarks following Proposition 2.1, there exists an onto Q-map φ such that

$$\varphi\colon S(B)+S(C)\to P(X).$$

Furthermore, there exist two sequences $Y_0, Y_1,\dots, Y_{m-1}$ and $Z_0,Z_1,\dots,Z_{n-1}$ of (not necessarily distinct) elements of $P_{\text{fin}}(\omega)$ such that B and C are isomorphic to subalgebras of $\prod(A_{Y_k}\colon k<m)$ and $\prod(A_{Z_k}\colon k<n)$, respectively. Hence, by the remarks following Proposition 2.1, there exist two onto Q-maps φ_B and φ_C such that

$$\varphi_B\colon\textstyle\sum(P(Y_k)\colon k<m)\to S(B)$$

and

$$\varphi_C\colon\textstyle\sum(P(Z_k)\colon k<n)\to S(C),$$

where $\sum(P(Y_k)\colon k<m)$ and $\sum(P(Z_k)\colon k<n)$ denote the Q-spaces obtained from the disjoint unions of the $P(Y_k)$'s and $P(Z_k)$'s, respectively.

Let

$$I_B=\{k\colon k<m \text{ and } \varphi\circ\varphi_B\restriction P(Y_k)(\{a,b\})=\{a,b\}\},$$

$$I_C=\{k\colon k<n \text{ and } \varphi\circ\varphi_C\restriction P(Z_k)(\{a,b\})=\{a,b\}\},$$

$$Y=\bigcup(Y_k:k\in I_B), \text{ and } Z=\bigcup(Z_k:k\in I_C).$$

We claim that $Y\cup Z=X$. By Lemma 5.3, $Y\cup Z\subseteq X$. To see that $X\subseteq Y\cup Z$, let $i\in X$ and choose $x\in V_i$. Since φ, φ_B, and φ_C are onto, $(x,3)$ is an element of $\varphi\circ\varphi_B(\sum(P(Y_k)\colon k<m)\cup\varphi\circ\varphi_C(\sum(P(Z_k)\colon k<n)$. If $(x,3)\in\varphi\circ\varphi_B(\sum(P(Y_k)\colon k<m)$, then, for some $k<m$, $(x,3)\in\varphi\circ\varphi_B(P(Y_k))$ which, by Lemma 5.2, implies $k\in I_B$. By Lemma 5.3, $(x,3)\in\varphi\circ\varphi_B(D_j)$ for some $j\in Y_k$. Since $(x,3)\in D_i$ and, by Lemma 5.3, $\varphi\circ\varphi_B(D_j)=D_j$, it follows that $i=j$. Thus, $i\in Y$. If $(x,3)\in\varphi\circ\varphi_C(\sum(P(Z_k)\colon k<n)$, then a similar argument shows that $i\in Z$. It follows that $X\subseteq Y\cup Z$ and, consequently, $X=Y\cup Z$, as claimed.

To complete the proof, it remains to show that $A_Y\in V_Q(\{B\})$ and $A_Z\in V_Q(\{C\})$. Since $X=Y\cup Z$ (by the preceding claim) and $X\neq\emptyset$, either Y or Z is nonempty. By Lemma 5.3, for $Y\neq\emptyset$.

$$P(Y)\subseteq\varphi\circ\varphi_B(\textstyle\sum(P(Y_k)\colon k<m))=\varphi(S(B)).$$

By Lemma 5.2, for $k<m$ with $k\notin I_B$,

$$\varphi\circ\varphi_B(P(Y_k))\subseteq P(Y)\cup\bigcup(V_i\times\{-k,k\}\colon i\in X\setminus Y \text{ and } k=1 \text{ or } 2)$$
$$\cup\bigcup(\{e_x,e_y\}\colon e=\{x,y\}\in E_i \text{ and } i\in X\setminus Y)$$

or

$$\varphi\circ\varphi_B(P(Y_k))\subseteq P(Y)\cup\bigcup(V_i\times\{-5,5\}\colon i\in X\setminus Y).$$

Hence,

$$P(Y)\subseteq\varphi(S(B))\subseteq P(Y)\cup\bigcup(V_i\times\{-k,k\}\colon i\in X\setminus Y \text{ and } k=1,2, \text{ or } 5)$$
$$\cup\bigcup(\{e_x,e_y\}\colon e=\{x,y\}\in E_i \text{ and } i\in X\setminus Y).$$

Choose an edge $e=\{y,z\}\in E_j$ for some $j\in Y$ and define a mapping $\psi_Y:\varphi(S(B))\to P(Y)$ by $\psi_Y((x,k))=(y,k)$ for $(x,k)\in\varphi(S(B))\setminus P(Y)$, $\psi_Y(\{e_u,e_v\})=\{e_y,e_z\}$ for $e=\{u,v\}$ with e_u and $e_v\in\varphi(S(B))\setminus P(Y)$, and ψ_Y is the identity on $P(Y)$. Clearly, ψ_Y is an onto Q-map. By the remarks following Proposition 2.1, it follows that A_Y is isomorphic to a subalgebra of B and so A_Y belongs to $V_Q(\{B\})$. A similar argument shows that $A_Z\in V_Q(\{C\})$ whenever $Z\neq\emptyset$. Thus, the family $(A_W\colon W\in P_{\mathrm{fin}}(\omega))$ satisfies (P4). □

By Proposition 4.1 and Lemmas 6.1–6.3, we conclude the following result.

Theorem 6.4. *The lattice $L(Q_{20})$ contains an isomorphic copy of the ideal lattice of a free lattice with ω free generators.*

Since the subvarieties of $\boldsymbol{Q}$ form a chain, for a variety $V \subseteq \boldsymbol{Q}$, either $V \subseteq \boldsymbol{Q}_{01}$ or $\boldsymbol{Q}_{20} \subseteq V$. Thus, Theorems 3.3 and 6.4 combine to establish the theorem of Section 1.

7. Related algebras

As indicated in Section 1, the class of Q-distributive lattices $\boldsymbol{Q}$ is related to the class $\boldsymbol{M}$ of monadic Boolean algebras. The algebras of $\boldsymbol{M}$ are obtained by endowing Boolean algebras with a unary operation ∇ satisfying the same postulates as those satisfied by ∇ on the algebras of $\boldsymbol{Q}$. The class $\boldsymbol{M}$ is a variety and the subvarieties of $\boldsymbol{M}$ form an $\omega+1$ chain $\boldsymbol{M}_0 \subset \boldsymbol{M}_1 \subset \boldsymbol{M}_2 \subset \cdots \subset \boldsymbol{M}$, where, for $p<\omega$, $\boldsymbol{M}_p$ is the variety generated by the p-atom Boolean algebra B_p where $\nabla 0=0$ and, for $a \neq 0$, $\nabla a=1$ (see [8]).

In this section, we show that the lattice structure of $L(\boldsymbol{M})$ is much simpler than the structure of $L(\boldsymbol{Q})$.

Let $P \subset \omega \times \omega$ denote the set consisting of all ordered pairs (i,j) such that $1 \leqslant i \leqslant j < \omega$. For $\preceq$ on P given by $(i,j) \preceq (k,l)$ if $i \leqslant k$ and $j \leqslant l$, $(P; \preceq)$ is a partially ordered set. For $1 \leqslant i < \omega$, let P_i denote the principal order ideal of P determined by (i,i). Let $D(P)$ and $D(P_i)$ denote the distributive lattices of all order ideals (including the empty set) of P and P_i, respectively, with set-theoretical union and intersection as the lattice operations.

Proposition 7.1. *$L(\boldsymbol{M}) \cong D(P)$ and, for $1 \leqslant i < \omega$, $L(\boldsymbol{M}_i) \cong D(P_i)$.*

Proof. Since $\boldsymbol{M}$ is locally finite, it is sufficient to consider the category $\boldsymbol{M}_{\text{fin}}$ of finite monadic Boolean algebras. For a Q-space $(P; \leqslant, E)$, the lattice $Q(P)$ is Boolean if and only if $(P; \leqslant)$ is an antichain. Thus, $\boldsymbol{M}_{\text{fin}}$ is dually equivalent to the full subcategory $\boldsymbol{T}_{\text{fin}}$ of $\boldsymbol{S}_{\text{fin}}$ whose objects $(P; \leqslant, E)$ are totally unordered partially ordered sets endowed with an equivalence relation. Let $X_0, \ldots, X_{n-1}$ denote the equivalence classes of a Q-space $(P; \leqslant, E)$ in $\boldsymbol{T}_{\text{fin}}$, where $|X_i| = m_i$ and $m_i \leqslant m_j$ for $0 \leqslant i \leqslant j < n$. Consider the Q-space $(P'; \leqslant \upharpoonright P', E \upharpoonright P')$ where $P' = X_0 \cup X_{n-1}$. Clearly, $(P'; \leqslant \upharpoonright P'. E \upharpoonright P')$ is a Q-map image of $(P; \leqslant, E)$ and $(P; \leqslant, E)$ is a Q-map image of $n-1$ copies of $(P'; \leqslant \upharpoonright P', E \upharpoonright P')$. Hence, $V_Q(\{Q(P)\}) = V_Q(\{Q(P')\})$. Furthermore, if $(P; \leqslant, E)$ and $((P_i; \leqslant, E_i): i<n)$ denote Q-spaces in $\boldsymbol{T}_{\text{fin}}$ each of which has precisely two equivalence classes, then $(P; \leqslant, E)$ is a Q-map image of $\sum((P_i; \leqslant, E_i): i<n)$ if and only if it is a Q-map image of $(P_i; \leqslant, E_i)$ for some $i<n$. The proof now follows from the remarks after Proposition 2.1. $\square$

The dual operation Δ to ∇, that is, the operation satisfying $\Delta 1=1$, $x \vee \Delta x = x$, $\Delta(x \wedge y) = \Delta x \wedge \Delta y$, and $\Delta(x \vee \Delta y) = \Delta x \vee \Delta y$, is not expressible by any term operation within $\boldsymbol{Q}$. (Clearly, within the class $\boldsymbol{M}$, $\Delta x = \neg \nabla \neg x$ where $\neg$ is Boolean complementation.) One may consider another class of algebras related to the algebras of $\boldsymbol{Q}$; namely, the class $\boldsymbol{Q}^+$ of algebras that result from the class of bounded distributive lattices by endowing them with both of the operations ∇ and Δ. We conclude by mentioning that the lattice $L(\boldsymbol{Q}^+)$ of quasivarieties contained in $\boldsymbol{Q}^+$ has similar properties to those of $L(\boldsymbol{Q})$. However, the lower part of $L(\boldsymbol{Q}^+)$ is more complex than that of $L(\boldsymbol{Q})$. Let $\boldsymbol{Q}_{01}^+$ denote the subvariety of $\boldsymbol{Q}^+$ consisting of all algebras whose Δ-free reducts belong to $\boldsymbol{Q}_{01}$. The variety $\boldsymbol{Q}_{01}^+$ is generated by the algebra Q_{01} endowed with the operation Δ where $\Delta 1 = 1$ and, for $a \neq 1$, $\Delta a = 0$. One may show that the ideal lattice of a free lattice with ω free generators is embeddable in $L(\boldsymbol{Q}_{01}^+)$.

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References

[1] M.E. Adams and W. Dziobiak, Q-universal quasivarieties of algebras, Proc. Amer. Math. Soc. 120 (1994) 1053–1059.
[2] R. Cignoli, Quantifiers on distributive lattices, Discrete Math. 96 (1991) 1991) 183–197.
[3] B.A. Davey and H.A. Priestley, Introduction to Lattices and Order (Cambridge University Press, Cambridge, 1990).
[4] G. Grätzer, H. Lakser, and R.W. Quackenbush, On the lattice of quasivarieties of distributive lattices with pseudocomplementation, Acta Sci. Math. (Szeged) 42 (1980) 257–263.
[5] P.R. Halmos, Algebraic logic, I. Monadic Boolean algebras, Compositio Math. 12 (1955) 217–249.
[6] Z. Hedrlin and J. Sichler, Any boundable binding category contains a proper class of mutually disjoint copies of itself, Algebra Universalis 1 (1971) 97–103.
[7] K.B. Lee, Equational classes of distributive pseudo-complemented lattices, Canad. J. Math. 22 (1970) 881–891.
[8] D. Monk, On equational classes of algebraic versions of logic. I, Math. Scand. 27 (1970) 53–71.
[9] H.A. Priestley, Representation of distributive lattices by means of ordered Stone spaces, Bull. London Math. Soc. 2 (1970) 186–190.
[10] H.A. Priestley, Natural dualities for varieties of distributive lattices with a quantifier, to appear.
[11] M.P. Tropin, An embedding of a free lattice into the lattice of quasivarieties of distributive lattices with pseudocomplementation, Algebra i Logika 22 (1983) 159–167 (in Russian); an English translation in: Algebra and Logic 22 (1983) 113–119.